AIAA JOURNAL
Vol. 43, No. 12, December 2005

New Model Correcting Method for Quadratic Eigenvalue Problems Using Symmetric Eigenstructure Assignment

Yuen-Cheng Kuo*
National Center for Theoretical Sciences, Hsinchu 300, Taiwan, Republic of China
Wen-Wei Lin†
National Tsinghua University, Hsinchu 300, Taiwan, Republic of China
and
Shu-Fang Xu‡
Peking University, 100871 Beijing, People's Republic of China

Finite element model correction of quadratic eigenvalue problems (QEPs) using a symmetric eigenstructure assignment technique was proposed by Zimmerman and Widengren (Zimmerman, D., and Widengren, M., "Correcting Finite Element Models Using a Symmetric Eigenstructure Assignment Technique," *AIAA Journal*, Vol. 28, No. 9, 1990, pp. 1670–1676) and incorporates the measured model data into the finite element model to produce an adjusted finite element model on the damping and stiffness matrices that matches the experimental model data and minimizes the distance between the analytical and corrected models. Slightly different from the cost function proposed by Zimmerman and Widengren, based on the penalty function given by Friswell et al. (Friswell, M. I., Inman, D. J., and Pilkey, D. F., "Direct Updating of Damping and Stiffness Matrices," *AIAA Journal*, Vol. 36, No. 3, 1998, pp. 491–493), a cost function is considered that which measures the distance between the analytical and corrected models in a least-squares sense. An efficient algorithm is developed to solve the corresponding optimization problem. The resulting matrices obtained by the new method are necessary and sufficient to the optimization problem. Furthermore, the computational cost of the proposed algorithm requires only $\mathcal{O}(nm^2)$ floating-point operations, where n is the size of coefficient matrices of the QEP and m is the number of the measured modes. The numerical results show that the new method is reliable and attractive.

Nomenclature

A	=	see Eq. (41a); $m \times m$
B	=	new control influence matrix, $n \times m$
B_0	=	actuator influence matrix, $n \times m$
$B(1:m,:)$	=	submatrix of B consisting of the 1–mth rows and all columns
$\boldsymbol{b}$	=	see Eq. (41b); $m \times 1$
C_0, C_1	=	output influence matrices, $r \times n$
D	=	adjusted damping matrix, $n \times n$
D_a	=	damping matrix of the original finite element method (FEM) model, $n \times n$
$D^{[4]}$	=	adjusted damping matrix from Ref. 4, $n \times n$
F	=	pseudocontrol matrix for derivatives, $m \times m$
G	=	pseudocontrol matrix for states, $m \times m$
$\imath$	=	$\sqrt{-1}$
K	=	adjusted stiffness matrix, $n \times n$
K_a	=	stiffness matrix of the original FEM model, $n \times n$
$K^{[4]}$	=	adjusted stiffness matrix from Ref. 4, $n \times n$
L	=	feedback gain matrix, $m \times r$
M	=	adjusted mass matrix, $n \times n$
M_a	=	mass matrix of the original FEM model, $n \times n$
m	=	number of control actuators, r
Q	=	orthogonal matrix, $n \times n$
q	=	weight factor
R	=	upper triangular matrix, $m \times m$
r	=	number of sensor outputs, r
S	=	$R\Lambda R^{-1}$, $m \times m$
$S\mathbb{R}^{m\times m}$	=	set of all symmetric matrices, $m \times m$
T	=	upper triangular matrix, $m \times m$
$\boldsymbol{u}$	=	control vector, $m \times 1$
W_{11}	=	see Eq. (25); $m \times m$
W_{21}	=	see Eq. (24); $(n-m) \times m$
$\boldsymbol{w}$	=	position vector, $n \times 1$
$\boldsymbol{x}$	=	solution of linear system, $m \times 1$
$\boldsymbol{y}$	=	output vector, $r \times 1$
$Z(1:m, j)$	=	jth subcolumn of matrix Z consisting of the first to mth components
$Z(:, j)$	=	jth column of matrix Z
Γ_j	=	see Eq. (38a); $m \times m$
Λ	=	diagonal block eigenvalue matrix; see Eq. (11a)
λ	=	eigenvalues of quadratic element problem (QEP)
Σ_j	=	see Eq. (38b); $m \times m$
Φ	=	eigenvector matrix; see Eq. (11b)
φ	=	eigenvectors of QEP, $n \times 1$
Ω	=	see Eq. (16); $m \times m$
∇	=	gradient of a function
$\|\ \|_F$	=	Frobenius norm

Superscripts

H	=	conjugate and transpose
T	=	transpose
$\cdot$	=	differentiation with respect to time

Received 23 February 2005; revision received 7 July 2005; accepted for publication 17 July 2005.

*Postdoctoral Fellow, Mathematics Division; m883207@am.nthu.edu.tw.
†Professor, Department of Mathematics; wwlin@am.nthu.edu.tw.
‡Professor, School of Mathematical Sciences; xsf@pku.edu.cn.

I. Introduction

A DISCRETIZATION of vibrating systems with feedback controls by the finite element method leads to an analytical second-order system

$$M_a\ddot{\boldsymbol{w}} + D_a\dot{\boldsymbol{w}} + K_a\boldsymbol{w} = B_0\boldsymbol{u} \tag{1}$$

where M_a, D_a, and $K_a \in \mathbb{R}^{n\times n}$ are all symmetric with M_a being symmetric positive definite ($M_a > 0$) and represent the mass, damping, and stiffness, respectively, $\boldsymbol{w}$ is the $n \times 1$ vector of positions,

B_0 is the $n \times m$ actuator influence matrix, $\boldsymbol{u}$ is the $m \times 1$ vector of control force, and the overdots represent differentiation with respect to time. In addition, the $r \times 1$ output vector $\boldsymbol{y}$ of the sensor measurement is given by

$$\boldsymbol{y} = C_0\boldsymbol{w} + C_1\dot{\boldsymbol{w}} \tag{2}$$

The control law is taken to be a general linear output feedback

$$\boldsymbol{u} = L\boldsymbol{y} \tag{3}$$

The finite element model in structured dynamics in detail may be found by Friswell and Mottershead.[1]

It is shown in Ref. 2 that if the system described by Eqs. (1) and (2) is controllable and observable, then by adequate selection of L, the closed-loop system

$$M_a\ddot{\boldsymbol{w}} + (D_a - B_0LC_1)\dot{\boldsymbol{w}} + (K_a - B_0LC_0)\boldsymbol{w} = 0 \tag{4}$$

has $\max(m, r)$ assigned eigenvalues and has $\max(m, r)$ partially assigned eigenvectors with $\min(m, r)$ entries in each eigenvector being arbitrarily assigned. Here, $\max(m, r)$ and $\min(m, r)$ take the maximal and minimal values between m and r, respectively.

Finite element model correction or updating has emerged in the 1990s as a significant subject for the design, construction, and maintenance of mechanical systems.[1,3] Finite element model correction of the closed-loop system (4) using a symmetric eigenstructure assignment was proposed in Refs. 4 and 5. The method incorporates the measured model data into the finite element model to produce an adjusted finite element model on damping and stiffness with symmetric low-rank updating that matches the experimental model data. With the equal numbers of pseudosensors and pseudoactuators, that is, $r = m$, the concept presented in Refs. 4 and 5 is to design a new influence matrix $B \in \mathbb{R}^{n\times m}$ and pseudocontrollers $F = F^T$ and $G = G^T \in \mathbb{R}^{m\times m}$ with the feedback controllers in Eq. (4) having the symmetric low-rank updating forms

$$B_0LC_1 = BFB^T, \qquad B_0LC_0 = BGB^T \tag{5}$$

such that the new closed-loop system

$$M\ddot{\boldsymbol{w}} + D\dot{\boldsymbol{w}} + K\boldsymbol{w} = 0 \tag{6a}$$

where

$$M = M_a > 0, \qquad D = D_a - BFB^T, \qquad K = K_a - BGB^T \tag{6b}$$

matches the partially assigned closed-loop eigenvalues and eigenvectors that are determined experimentally.

To solve the homogeneous second-order system Eq. (6) is known as to solve the quadratic eigenvalue problem (QEP),

$$(\lambda^2M + \lambda D + K)\varphi = 0 \tag{7}$$

by letting $\boldsymbol{w} = e^{\lambda t}\varphi$ in Eq. (6a), where $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{C}^n$ are eigenvalues and eigenvectors of QEP, respectively.

Based on the concept in Ref. 4, we now formulate the inverse QEPs associated with Eqs. (5) and (6) as follows.

Problem 1: Given $M_a = M_a^T > 0$, $D_a = D_a^T$, $K_a = K_a^T$ and the measured eigenmatrix pair $(\Lambda, \Phi) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m}$, find $B \in \mathbb{R}^{n\times m}$, $F = F^T$, and $G = G^T \in \mathbb{R}^{m\times m}$ such that

$$M\Phi\Lambda^2 + D\Phi\Lambda + K\Phi = 0 \tag{8}$$

where $M = M_a$, $D = D_a - BFB^T$, and $K = K_a - BGB^T$.

Note that other studies in the works by Datta,[6] Datta et al.,[7] Datta and Sarkissian,[8] and Lin and Wang[9] lead to a partial eigenstructure assignment for the QEP which is taken as a nonsymmetric feedback design problem for the second-order control system.

Problem 2: Let

$$\mathbb{S} = \{(B, F, G) \in \mathbb{R}^{n\times m} \times S\mathbb{R}^{m\times m} \times S\mathbb{R}^{m\times m} | (B, F, G) \text{ solves problem 1}\}$$

and $q = \|D_a\|_F/\|K_a\|_F$. Solve the optimization problem

$$\min\left\{J_2 = \left\|BFB^T\right\|_F^2 + q^2\left\|BGB^T\right\|_F^2 \middle| (B, F, G) \in \mathbb{S}\right\} \tag{9}$$

Remark: The cost function J_2 of problem (9) in a least-squares sense is slightly different from the cost function $J_1 = \|BFB^T\|_F + q\|BGB^T\|_F$ proposed in Ref. 4, which did not square the norms of stiffness and damping perturbations. However, as shown in Ref. 4, the optimization problem for J_1 cannot be solved explicitly. To find an explicit form of the unique solution for the optimization problem (9), we borrow the penalty function in Ref. 10 that which is given by

$$J = \|D - D_a\|_F^2 + \mu\|K - K_a\|_F^2 \tag{10}$$

Note that the optimums of $\min\{J_1\}$ and $\min\{J_2\}$ are not necessarily equal; however, from mathematical viewpoint, two measures, J_1 and J_2, are equivalent, that is, there are constants $c > 0$ and $d > 0$ such that $cJ_1 \le \sqrt{J_2} \le dJ_1$.

The methods proposed by Friswell et al.[10] and improved by Kuo et al.[11] are to find symmetric damping D and stiffness K that minimize the penalty function J in Eq. (10), and satisfy Eq. (8). Zimmerman and Widengren[4] and Zimmerman and Kaouk[12] developed algorithms to solve problems 1 and 2 that require solving a generalized algebraic Riccati equation and several linear systems. However, the computed F and G are only the necessary but not sufficient condition for the symmetric low-rank correction in Eq. (5). The purpose of this paper is to develop efficient algorithms to solve problems 1 and 2 that requires only $\mathcal{O}(nm^2)$ floating-point operations (FLOPS). Furthermore, the resulting matrices obtained by the new method are necessary and sufficient to problems 1 and 2.

II. Solving a Partially Described Inverse QEP

To match the partial measured data of the spectrum information of a QEP, we consider solving the partially described inverse QEP (PD-IQEP).

Let $(\Lambda, \Phi) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m} (m \le n)$ be a given pair of matrices, where

$$\Lambda = \mathrm{diag}\left(\lambda_1^{[2]}, \ldots, \lambda_\ell^{[2]}, \lambda_{2\ell+1}, \ldots, \lambda_m\right) \tag{11a}$$

with

$$\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}, \qquad \beta_j \ne 0$$

for $j = 1, \ldots, \ell$, and

$$\Phi = [\varphi_{1R}, \varphi_{1I}, \ldots, \varphi_{\ell R}, \varphi_{\ell I}, \varphi_{2\ell+1}, \ldots, \varphi_m] \tag{11b}$$

Note that the eigenvalues of $\lambda_j^{[2]}$ are just the complex conjugate pair $\alpha_j \pm \beta_j\imath$ $(\imath = \sqrt{-1})$. Suppose that Λ has only simple eigenvalues and Φ is of full column rank. Find a general form of symmetric matrices M, D, and K, with M being symmetric positive definite, that satisfy the equation

$$M\Phi\Lambda^2 + D\Phi\Lambda + K\Phi = 0 \tag{12}$$

Let Φ have the QR factorization

$$\Phi = Q\begin{bmatrix} R \\ 0 \end{bmatrix} \equiv [Q_1, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix} \tag{13}$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal with $Q_1 \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$ is nonsingular. Partition Q^TMQ, Q^TDQ, and Q^TKQ by

$$Q^TMQ = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad Q^TDQ = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

$$Q^TKQ = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{14}$$

where M_{11}, D_{11}, and $K_{11} \in \mathbb{R}^{m\times m}$.

A general solution of symmetric M, D, and K, with M being symmetric positive definite, is given by the theorem in Ref. 13.

Theorem 1 (Ref. 13): For a given matrix pair (Λ, Φ) as in Eq. (11), the general solution of PD-IQEP is given by

$$M_{11} : \text{arbitrary fixed symmetric positive definite matrix} \quad (15a)$$

$$D_{11} = -\left(M_{11}S + S^TM_{11} + R^{-T}\Omega R^{-1}\right) \quad (15b)$$

$$K_{11} = S^TM_{11}S + R^{-T}\Omega\Lambda R^{-1} \quad (15c)$$

$$K_{21} = K_{12}^T = -\left(M_{21}S^2 + D_{21}S\right) \quad (15d)$$

where $S = R\Lambda R^{-1}$ and

$$\Omega = \operatorname{diag}\left(\begin{bmatrix}\xi_1 & \eta_1\\ \eta_1 & -\xi_1\end{bmatrix}, \ldots, \begin{bmatrix}\xi_\ell & \eta_\ell\\ \eta_\ell & -\xi_\ell\end{bmatrix}, \xi_{2\ell+1}, \ldots, \xi_m\right) \quad (16)$$

in which ξ_i and η_i are arbitrary real numbers. Furthermore, $M_{21} = M_{12}^T$, $D_{21} = D_{12}^T$, $D_{22} = D_{22}^T$, and $K_{22} = K_{22}^T$ are chosen arbitrarily, and $M_{22} = M_{22}^T$ is chosen so that $M_{22} - M_{21}M_{11}^{-1}M_{12}$ is symmetric positive definite.

III. Solving Problem 1

In this section, we shall develop an efficient algorithm for solving problem 1 described in the Introduction. Let $(\Lambda, \Phi) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m}$ be given in Eq. (11) with QR factorization of Φ in Eq. (13). According to the partition of Eq. (14) and letting

$$\hat{B} = Q^TB = \begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}, \qquad \hat{B}_1 \in \mathbb{R}^{m\times m} \quad (17)$$

Eqs. (6b) can be written as

$$Q^TDQ = Q^TD_aQ - \hat{B}F\hat{B}^T \quad (18)$$

$$Q^TKQ = Q^TK_aQ - \hat{B}G\hat{B}^T \quad (19)$$

By Theorem 1, and from Eqs. (13–15) and (17), Eqs. (18) and (19), respectively, become

$$\begin{bmatrix}-M_{11}S - S^TM_{11} - R^{-T}\Omega R^{-1} & D_{21}^T\\ D_{21} & D_{22}\end{bmatrix} = \begin{bmatrix}Q_1^TD_aQ_1 & Q_1^TD_aQ_2\\ Q_2^TD_aQ_1 & Q_2^TD_aQ_2\end{bmatrix} - \begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}F\left[\hat{B}_1^T, \hat{B}_2^T\right] \quad (20)$$

$$\begin{bmatrix}S^TM_{11}S + R^{-T}\Omega\Lambda R^{-1} & -\left(M_{21}S^2 + D_{21}S\right)^T\\ -\left(M_{21}S^2 + D_{21}S\right) & K_{22}\end{bmatrix} = \begin{bmatrix}Q_1^TK_aQ_1 & Q_1^TK_aQ_2\\ Q_2^TK_aQ_1 & Q_2^TK_aQ_2\end{bmatrix} - \begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}G\left[\hat{B}_1^T, \hat{B}_2^T\right] \quad (21)$$

where $M_{11} = Q_1^TM_aQ_1$ and $M_{21} = Q_2^TM_aQ_1$. Comparing blocks in Eqs. (20) and (21), respectively, we get

$$\hat{B}_1F\hat{B}_1^T = M_{11}S + S^TM_{11} + R^{-T}\Omega R^{-1} + Q_1^TD_aQ_1 \quad (22a)$$

$$\hat{B}_2F\hat{B}_1^T = -D_{21} + Q_2^TD_aQ_1 \quad (22b)$$

$$\hat{B}_2F\hat{B}_2^T = -D_{22} + Q_2^TD_aQ_2 \quad (22c)$$

$$\hat{B}_1G\hat{B}_1^T = -S^TM_{11}S - R^{-T}\Omega\Lambda R^{-1} + Q_1^TK_aQ_1 \quad (23a)$$

$$\hat{B}_2G\hat{B}_1^T = M_{21}S^2 + D_{21}S + Q_2^TK_aQ_1 \quad (23b)$$

$$\hat{B}_2G\hat{B}_2^T = -K_{22} + Q_2^TK_aQ_2 \quad (23c)$$

Substituting Eq. (22b) into Eq. (23b), we have

$$\hat{B}_2\left(G\hat{B}_1^T + F\hat{B}_1^TS\right) = M_{21}S^2 + \left(Q_2^TD_aQ_1\right)S + Q_2^TK_aQ_1 \equiv W_{21} \quad (24)$$

Multiplying Eq. (22a) by S from the right and adding the result to Eq. (23a), we get

$$\hat{B}_1\left(G\hat{B}_1^T + F\hat{B}_1^TS\right) = M_{11}S^2 + \left(Q_1^TD_aQ_1\right)S + Q_1^TK_aQ_1 \equiv W_{11} \quad (25)$$

Hereafter, we assume that W_{11} is nonsingular. It is easy to derive that

$$W_{11} = R^{-T}\left[\Phi^T\left(M_a\Phi\Lambda^2 + D_a\Phi\Lambda + K_a\Phi\right)\right]R^{-1}$$

and hence, requiring W_{11} to be nonsingular amounts to requiring $\Phi^T(M_a\Phi\Lambda^2 + D_a\Phi\Lambda + K_a\Phi)$ to be nonsingular $(\mathcal{H})$. It follows from Eq. (25) that $\hat{B}_1$ is invertible, and hence, we have

$$G\hat{B}_1^T + F\hat{B}_1^TS = \hat{B}_1^{-1}W_{11} \quad (26)$$

Substituting Eq. (26) into Eq. (24), we get

$$\hat{B}_2 = W_{21}W_{11}^{-1}\hat{B}_1 \quad (27)$$

This, together with Eqs. (22a) and (23a), gives rise to the following result.

Theorem 2: Given $M_a = M_a^T > 0$, $D_a = D_a^T$, $K_a = K_a^T$, and the eigenmatrix pair $(\Lambda, \Phi) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m}$, assume that $(\mathcal{H})$ holds. Then problem 1 is solvable with

$$B = Q\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix} \in \mathbb{R}^{n\times m}$$

$$F = \hat{B}_1^{-1}\left(M_{11}S + S^TM_{11} + R^{-T}\Omega R^{-1} + Q_1^TD_aQ_1\right)\hat{B}_1^{-T} \in \mathbb{R}^{m\times m}$$

$$G = \hat{B}_1^{-1}\left(-S^TM_{11}S - R^{-T}\Omega\Lambda R^{-1} + Q_1^TK_aQ_1\right)\hat{B}_1^{-T} \in \mathbb{R}^{m\times m}$$

where $\hat{B}_1 \in \mathbb{R}^{m\times m}$ is arbitrary and nonsingular, $\hat{B}_2$ is given by Eq. (27), W_{21} and W_{11} are given in Eqs. (24) and (25), respectively, and Ω is given by Eq. (16).

IV. Solving Problem 2

Assume that $(\mathcal{H})$ holds, Let $(B, F, G) \in \mathbb{S}$ and let $J = \|BFB^T\|_F^2 + q^2\|BGB^T\|_F^2$, where $q = \|D_a\|_F/\|K_a\|_F$. Then, by Theorem 2, it follows that $\hat{B}_2\hat{B}_1^{-1} = W_{21}W_{11}^{-1} \equiv W$, which is independent of the choice of $\hat{B}_1$, and hence,

$$J = \left\|\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}F\left[\hat{B}_1^T, \hat{B}_2^T\right]\right\|_F^2 + q^2\left\|\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}G\left[\hat{B}_1^T, \hat{B}_2^T\right]\right\|_F^2 = \left\|\begin{bmatrix}I\\ W\end{bmatrix}\hat{F}[I, W^T]\right\|_F^2 + q^2\left\|\begin{bmatrix}I\\ W\end{bmatrix}\hat{G}[I, W^T]\right\|_F^2 \quad (28)$$

is independent of the choice of $\hat{B}_1$, where

$$\hat{F} = M_{11}S + S^TM_{11} + R^{-T}\Omega R^{-1} + Q_1^TD_aQ_1 \quad (29a)$$

$$\hat{G} = -S^TM_{11}S - R^{-T}\Omega\Lambda R^{-1} + Q_1^TK_aQ_1 \quad (29b)$$

Let $\hat{Q}_2 \in \mathbb{R}^{(n-m)\times(n-m)}$ be an orthogonal matrix, such that

$$\hat{Q}_2WR^{-T} = \begin{bmatrix}T\\ 0\end{bmatrix} \quad (30)$$

where T is upper triangular. Define $\hat{Q}_2 W = \hat{W}$ and

$$\begin{bmatrix} I \\ \hat{W} \end{bmatrix} \left(M_{11}S + S^T M_{11} + Q_1^T D_a Q_1\right)[I, \hat{W}^T] \equiv \begin{bmatrix} F_{11} & F_{21}^T \\ F_{21} & F_{22} \end{bmatrix} \tag{31a}$$

$$\begin{bmatrix} I \\ \hat{W} \end{bmatrix} \left(S^T M_{11} S - Q_1^T K_a Q_1\right)[I, \hat{W}^T] \equiv \begin{bmatrix} G_{11} & G_{21}^T \\ G_{21} & G_{22} \end{bmatrix} \tag{31b}$$

Then, the optimization problem (9) is equivalent to

$$\begin{aligned} \min = {} & \left\| F_{11} + R^{-T}\Omega R^{-T} \right\|_F^2 + q^2 \left\| G_{11} + R^{-T}\Omega\Lambda R^{-1} \right\|_F^2 \\ & + 2\left\| F_{21} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Omega R^{-1} \right\|_F^2 + 2q^2 \left\| G_{21} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Omega\Lambda R^{-1} \right\|_F^2 \\ & + \left\| F_{22} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Omega[T^T, 0] \right\|_F^2 + q^2 \left\| G_{22} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Omega\Lambda[T^T, 0] \right\|_F^2 \end{aligned} \tag{32}$$

where Ω is the undetermined block diagonal as in Eq. (16). Let

$$\boldsymbol{x} = [\xi_1, \eta_1, \ldots, \xi_\ell, \eta_\ell, \xi_{2\ell+1}, \ldots, \xi_m]^T \tag{33}$$

corresponding to the matrix Ω in Eq. (16). Then, Eq. (32) can be rewritten as

$$\min = f(\boldsymbol{x}) + 2g(\boldsymbol{x}) + h(\boldsymbol{x}) \tag{34}$$

where

$$f(\boldsymbol{x}) = \left\| F_{11} + R^{-T}\Omega R^{-1} \right\|_F^2 + q^2 \left\| G_{11} + R^{-T}\Lambda^T\Omega R^{-1} \right\|_F^2 \tag{35a}$$

$$g(\boldsymbol{x}) = \left\| F_{21} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Omega R^{-1} \right\|_F^2 + q^2 \left\| G_{21} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Lambda^T\Omega R^{-1} \right\|_F^2 \tag{35b}$$

$$\begin{aligned} h(\boldsymbol{x}) = {} & \left\| F_{22} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Omega[T^T, 0] \right\|_F^2 \\ & + q^2 \left\| G_{22} + \begin{bmatrix} T \\ 0 \end{bmatrix} \Lambda^T\Omega[T^T, 0] \right\|_F^2 \end{aligned} \tag{35c}$$

Let

$$R^{-1} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m] = \begin{bmatrix} r_{11} & \ldots & r_{1m} \\ & \ddots & \vdots \\ 0 & & r_{mm} \end{bmatrix} \tag{36a}$$

$$T^T = [\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m] = \begin{bmatrix} t_{11} & & 0 \\ \vdots & \ddots & \\ t_{m1} & \ldots & t_{mm} \end{bmatrix} \tag{36b}$$

From Eqs. (33) and (36), it is easily seen that the vectors $\Omega\boldsymbol{r}_j$ and $\Omega\boldsymbol{t}_j$ can be, respectively, rewritten as

$$\Omega\boldsymbol{r}_j = \Gamma_j\boldsymbol{x}, \qquad \Omega\boldsymbol{t}_j = \Sigma_j\boldsymbol{x} \tag{37}$$

for $j = 1, \ldots, m$, where

$$\Gamma_j = \operatorname{diag}\left(\begin{bmatrix} r_{1,j} & r_{2,j} \\ -r_{2,j} & r_{1,j} \end{bmatrix}, \ldots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j} \end{bmatrix}, r_{2\ell+1,j}, \ldots, r_{m,j} \right) \tag{38a}$$

$$\Sigma_j = \operatorname{diag}\left(\begin{bmatrix} t_{1,j} & t_{2,j} \\ -t_{2,j} & t_{1,j} \end{bmatrix}, \ldots, \begin{bmatrix} t_{2\ell-1,j} & t_{2\ell,j} \\ -t_{2\ell,j} & t_{2\ell-1,j} \end{bmatrix}, t_{2\ell+1,j}, \ldots, t_{m,j} \right) \tag{38b}$$

Substituting Eq. (37) into Eq. (35), we get

$$\begin{aligned} \nabla f(\boldsymbol{x}) = 2\sum_{j=1}^{m} \Big[& \left(R^{-T}\Gamma_j\right)^T F_{11}(:, j) + \Gamma_j^T R^{-1} R^{-T}\Gamma_j\boldsymbol{x} \\ & + q^2\Gamma_j^T\Lambda R^{-1} G_{11}(:, j) + q^2\Gamma_j^T\Lambda R^{-1}R^{-T}\Lambda^T\Gamma_j\boldsymbol{x} \Big] \end{aligned} \tag{39a}$$

$$\begin{aligned} \nabla g(\boldsymbol{x}) = 2\sum_{j=1}^{m} \Big[& (T\Gamma_j)^T F_{21}(1:m, j) + \Gamma_j^T T^T T\Gamma_j\boldsymbol{x} \\ & + q^2\Gamma_j^T\Lambda T^T G_{21}(1:m, j) + q^2\Gamma_j^T\Lambda T^T T\Lambda^T\Gamma_j\boldsymbol{x} \Big] \end{aligned} \tag{39b}$$

$$\begin{aligned} \nabla h(\boldsymbol{x}) = 2\sum_{j=1}^{m} \Big[& (T\Sigma_j)^T F_{22}(1:m, j) + \Sigma_j^T T^T T\Sigma_j\boldsymbol{x} \\ & + q^2\Sigma_j^T\Lambda T^T G_{22}(1:m, j) + q^2\Sigma_j^T\Lambda T^T T\Lambda^T\Sigma_j\boldsymbol{x} \Big] \end{aligned} \tag{39c}$$

Setting $\nabla[f(\boldsymbol{x}) + 2g(\boldsymbol{x}) + h(\boldsymbol{x})] = 0$, we derive the linear equation for $\boldsymbol{x}$

$$A\boldsymbol{x} = \boldsymbol{b} \tag{40}$$

where

$$\begin{aligned} A = {} & \sum_{j=1}^{m} \left(\Gamma_j^T R^{-1}R^{-T}\Gamma_j + 2\Gamma_j^T T^T T\Gamma_j + \Sigma_j^T T^T T\Sigma_j \right) \\ & + q^2\sum_{j=1}^{m} \Big(\Gamma_j^T\Lambda R^{-1}R^{-T}\Lambda^T\Gamma_j + 2\Gamma_j^T\Lambda T^T T\Lambda^T\Gamma_j \\ & + \Sigma_j^T\Lambda T^T T\Lambda^T\Sigma_j \Big) \end{aligned} \tag{41a}$$

$$\begin{aligned} \boldsymbol{b} = -\sum_{j=1}^{m} \Big[& \left(R^{-T}\Gamma_j\right)^T F_{11}(:, j) + 2(T\Gamma_j)^T F_{21}(1:m, j) \\ & + (T\Sigma_j)^T F_{22}(1:m, j) \Big] - q^2\sum_{j=1}^{m} \Big[\Gamma_j^T\Lambda R^{-1}G_{11}(:, j) \\ & + 2\Gamma_j^T\Lambda T^T G_{21}(1:m, j) + \Sigma_j^T\Lambda T^T G_{22}(1:m, j) \Big] \end{aligned} \tag{41b}$$

The steps for solving problem 2 are summarized in the following algorithm.

Algorithm: The input is $M_a = M_a^T > 0$, $D_a = D_a^T$, $K_a = K_a^T$, and $(\Lambda, \Phi) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m}$ as in Eq. (11). The output is $B \in \mathbb{R}^{n\times m}$, $F = F^T$, and $G = G^T \in \mathbb{R}^{m\times m}$ so that (B, F, G) solves the optimization problem (9) and satisfies Eq. (8) with $M_a = M_a^T > 0$, $D = D_a - BFB^T$, and $K = K_a - BGB^T$.

1) Compute the QR factorization of Φ,

$$\Phi = Q\begin{bmatrix} R \\ 0 \end{bmatrix} \equiv [Q_1, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}, \qquad S = R\Lambda R^{-1}$$

2) Compute $M_{11} = Q_1^T M_a Q_1$ and $M_{21} = Q_2^T M_a Q_1$.

3) Compute W_{21} and W_{11} by Eqs. (24) and (25), respectively, and $W = W_{21}W_{11}^{-1}$.

4) Compute the QR factorization of WR^{-T},

$$\hat{Q}_2(WR^{-T}) = \begin{bmatrix} T \\ 0 \end{bmatrix}, \qquad \hat{W} = \hat{Q}_2 W$$

5) Compute F_{ij} and G_{ij}, $i, j = 1, 2$, as in Eq. (31), and compute r_{ij} and t_{ij}, $i, j = 1, \ldots, m$, as in Eq. (36).

6) Solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A and $\boldsymbol{b}$ are evaluated by Eq. (41), in which Γ_j and Σ_j, $j = 1, \ldots, m$, are given by Eq. (38).

7) Construct Ω as in Eq. (16) by $\boldsymbol{x}$ of the form of Eq. (33), and compute

$$B = Q\begin{bmatrix} I \\ W \end{bmatrix}$$

$$F = M_{11}S + S^T M_{11} + R^{-T}\Omega R^{-1} + Q_1^T D_a Q_1$$

$$G = -S^T M_{11} S - R^{-T}\Omega\Lambda R^{-1} + Q_1^T K_a Q_1$$

8) Compute $M_a = M_a^T$, $D = D_a - BFB^T$, and $K = K_a - BGB^T$.

V. Numerical Results

We test examples provided in Ref. 4 using the algorithm to solve optimization problem (9). Consider an analytical five-degree-of-freedom system with mass, damping, and stiffness matrices given by

$$M_a = \text{diag}(1, 2, 5, 4, 3)$$

$$D_a = \begin{bmatrix} 11 & -2 & 0 & 0 & 0 \\ & 14 & -3.5 & 0 & 0 \\ & & 13.0 & -1.2 & 0 \\ & \text{sym.} & & 13.5 & -4 \\ & & & & 15.4 \end{bmatrix}$$

$$K_a = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ & 120 & -35 & 0 & 0 \\ & & 80 & -12 & 0 \\ & \text{sym.} & & 95 & -40 \\ & & & & 124 \end{bmatrix}$$

The model used to simulate the consistent experimental data is given by $M_e = M_a$, $D_e = D_a$, and

$$K_e = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ & 120 & -35 & 0 & 0 \\ & & 70 & -12 & 0 \\ & \text{sym} & & 95 & -40 \\ & & & & 124 \end{bmatrix}$$

Note that the difference between K_a and K_e is in the (3,3) element. The eigensolution of the experimental model is used to create the experimental model data. It is assumed that only the fundamental mode characteristics are experimentally determined and the only the second and third components of the eigenvector are measured:

$$\Lambda = \lambda_1^{[2]} = \begin{bmatrix} -1.116 & 3.057 \\ -3.057 & -1.116 \end{bmatrix}$$

$$\text{or} \qquad \lambda_1 = -1.116 + 3.057\imath\left(\imath = \sqrt{-1}\right) \tag{42a}$$

$$\tilde{\Phi} = [\tilde{\varphi}_{1R}, \tilde{\varphi}_{1I}] = \begin{bmatrix} \star & 0.3708 & 1 & \star & \star \\ \star & 0.0048 & 0 & \star & \star \end{bmatrix}^T \tag{42b}$$

where the $\star$ represents the unmeasured components of the eigenvector.

Test 1

The undetermined components $\star$ in Eq. (42b) are set to zero,[4] that is,

$$\Phi = \begin{bmatrix} 0 & 0.3708 & 1 & 0 & 0 \\ 0 & 0.0048 & 0 & 0 & 0 \end{bmatrix}^T \tag{43}$$

The algorithm computes matrices B, F, and G with

$$B = \begin{bmatrix} 1.5335 & -0.3477 & -0.9376 & 2.3893 & 0 \\ 4.1022 & -0.9376 & 0.3477 & 6.2589 & 0 \end{bmatrix}^T$$

$$F = \begin{bmatrix} -0.4886 & 0.0068 \\ 0.0068 & 0.0629 \end{bmatrix}, \qquad G = \begin{bmatrix} -12.905 & 3.1287 \\ 3.1287 & -0.5027 \end{bmatrix} \tag{44}$$

which solve the optimization problem (9). The resulting corrected finite element model $(M, D, K) \equiv (M_a, D_a - BFB^T, K_a - BGB^T)$ satisfies the error bound

$$\left\| M_a\Phi\Lambda^2 + D\Phi\Lambda + K\Phi \right\|_F = 1.4312 \times 10^{-11} \tag{45}$$

Test 2

Choose

$$B_0 = \begin{bmatrix} 0.0828 & 0.2878 & 2.0158 & 0.6993 & 0.2406 \\ 0.0353 & 0.3097 & 2.0114 & 0.6995 & 0.2406 \end{bmatrix}^T$$

as in Ref. 4. Let Π_1 be the permutation that reorders the measured components of $\tilde{\varphi}_{1R} + \imath\tilde{\varphi}_{1I}$ to the top portion of vector, that is,

$$\Pi_1(\tilde{\varphi}_{1R} + \imath\tilde{\varphi}_{1I}) = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{f}_1 \end{bmatrix} \tag{46}$$

where $\boldsymbol{u}_1 \in \mathbb{C}^s$ is the measured $s \times 1$ eigenvector and $\boldsymbol{f}_1 \in \mathbb{C}^{n-s}$ is the vector of free entries. Here $s = 2$. Compute

$$L_1 = \left(\lambda_1^2 M_a + \lambda_1 D_a + K_a\right)^{-1} B_0 \tag{47a}$$

$$\Pi_1 L_1 = \begin{bmatrix} \hat{L}_1 \\ D_1 \end{bmatrix} \qquad \text{with} \qquad \hat{L}_1 \in \mathbb{C}^{s\times s} \tag{47b}$$

The best achievable eigenvector, in a least-squares sense, is then given by

$$\varphi_{1R} + \imath\varphi_{1I} = L_1\left(\hat{L}_1^H \hat{L}_1\right)^{-1} \hat{L}_1^H \boldsymbol{u}_1 \tag{48}$$

With $\Lambda = \lambda_1^{[2]}$ and $\Phi = [\varphi_{1R}, \varphi_{1I}]$, the algorithm computes

$$B = 10^{-2} \times \begin{bmatrix} -7.7719 & -11.826 & -95.645 & -33.040 & -11.373 \\ 92.358 & -43.399 & 3.5237 & -2.1419 & -0.6029 \end{bmatrix}^T$$

$$F = 10^{-3} \times \begin{bmatrix} -3.5842 & -2.7037 \\ -2.7037 & -9.8095 \end{bmatrix}, \qquad G = \begin{bmatrix} -7.7269 & 4.6082 \\ 4.6082 & 0.4854 \end{bmatrix} \tag{49}$$

which solves optimization problem (9). The resulting corrected damping and stiffness matrices, $D = D_a - BFB^T$ and $K = K_a - BGB^T$, are given by

$$D = \begin{bmatrix} 10.99 & -2.00 & 0.00 & 0.00 & 0.00 \\ & 14.00 & -3.50 & 0.00 & 0.00 \\ & & 13.00 & -1.20 & 0.00 \\ & \text{sym.} & & 13.50 & -4.00 \\ & & & & 15.40 \end{bmatrix}$$

$$K = \begin{bmatrix} 99.71 & -20.61 & -4.64 & -1.61 & -0.55 \\ & 120.5 & -34.0 & 0.38 & 0.13 \\ & & 72.62 & -14.4 & -0.83 \\ & \text{sym} & & 94.22 & -40.3 \\ & & & & 123.9 \end{bmatrix}$$

When the matrix $B \times B(1:2,:)^{-1}$ is compared with the influence matrix $B_0 \times B_0(1:2,:)^{-1}$, it has 12 correcting digits. The optimal value of optimization problem (9) is 1.8911, and the error bound of the residual is estimated by

$$\left\| M_a \Phi \Lambda^2 + D \Phi \Lambda + K \Phi \right\|_F = 9.6988 \times 10^{-13} \tag{50}$$

Test 3

Choose

$$B_0 = \begin{bmatrix} 0.0397 & 0.3561 & 3.2744 & 0.4810 & 0.2127 \\ 0.0429 & 0.3635 & -0.8633 & 0.4881 & 0.2170 \end{bmatrix}^T$$

as in Ref. 4. We compute a new $\varphi_{1R} + \imath\varphi_{1I}$ by Eqs. (46–48). With $\Lambda = \lambda_1^{[2]}$ and $\Phi = [\varphi_{1R}, \varphi_{1I}]$, the algorithm computes

$$B = 10^{-2} \times \begin{bmatrix} -2.6649 & -23.283 & -91.648 & -31.365 & -13.904 \\ -6.7395 & -57.505 & 52.988 & -77.274 & -34.332 \end{bmatrix}^T$$

$$F = 10^{-3} \times \begin{bmatrix} 2.3941 & -12.648 \\ -12.648 & -8.0685 \end{bmatrix}, \qquad G = \begin{bmatrix} -7.8219 & 3.1530 \\ 3.1530 & 0.3358 \end{bmatrix} \tag{51}$$

The resulting corrected damping and stiffness matrices are given by

$$D = \begin{bmatrix} 11.00 & -2.00 & 0.00 & 0.00 & 0.00 \\ & 13.99 & -3.50 & -0.01 & -0.00 \\ & & 13.01 & -1.20 & 0.00 \\ & \text{sym} & & 13.49 & -4.00 \\ & & & & 15.40 \end{bmatrix}$$

$$K = \begin{bmatrix} 100.0 & -19.94 & -0.05 & 0.08 & 0.04 \\ & 120.5 & -35.5 & 0.71 & 0.32 \\ & & 70.46 & -12.7 & -0.30 \\ & \text{sym} & & 95.96 & -39.6 \\ & & & & 124.2 \end{bmatrix}$$

The resulting matrix $B \times B(1:2,:)^{-1}$ has 12 correcting digits as does the influence matrix $B_0 \times B_0(1:2,:)^{-1}$. The error bound of the residual is estimated by

$$\left\| M_a \Phi \Lambda^2 + D \Phi \Lambda + K \Phi \right\|_F = 7.4647 \times 10^{-13} \tag{52}$$

The optimal value of optimization problem (9), in a least-squares sense, is 1.4991 and the value of $J_2 = \|D^{[4]} - D_a\|_F^2 + q^2\|K^{[4]} - K_a\|_F^2$ is 1.8842, where $D^{[4]}$ and $K^{[4]}$ satisfy the optimal value of $J_1 = \|D^{[4]} - D_a\|_F + q\|K^{[4]} - K_a\|_F$, which are obtained in Ref. 4.

Test 4

In this test, we study the effects of noise on the updating process. We consider to perturb 1) the eigenvalue matrix $\lambda_1^{[2]}$ in Eq. (42a) by

$$\tilde{\lambda}_1^{[2]} = \lambda_1^{[2]} + 10^{-4} \begin{bmatrix} \xi & \eta \\ -\eta & \xi \end{bmatrix}$$

with $\xi = \text{rand}(1)$ and $\eta = \text{rand}(1)$, where rand(1) is a randomly generated number; and 2) the eigenvector matrix Φ in Eq. (43) by $\tilde{\Phi} = \Phi + 10^{-4} \times \text{rand}(2, 5)$, where rand(2, 5) is a randomly generated 2×5 matrix. The algorithm is used to solve optimization problem (9) with 1000 randomly chosen eigenvalue/eigenvector matrices of cases 1 and 2, respectively. The differences between the resulting corrected finite element models $\tilde{D}$, $\tilde{K}$ and finite element models D, K obtained by test 1 with unperturbed $\lambda_1^{[2]}$ and Φ are, respectively, presented as follows.

Case 1:

$$2 \times 10^{-5} \le \|\tilde{D} - D\|_F \le 8 \times 10^{-4}$$

$$5 \times 10^{-4} \le \|\tilde{K} - K\|_F \le 3.5 \times 10^{-3} \tag{53}$$

Case 2:

$$2 \times 10^{-3} \le \|\tilde{D} - D\|_F \le 9 \times 10^{-3}$$

$$10^{-2} \le \|\tilde{K} - K\|_F \le 6 \times 10^{-2} \tag{54}$$

From inequalities (53) and (54) we observe that for case 1 the order of perturbation between $\tilde{D}$ and D, as well as $\tilde{K}$ and K, is the same as the order of noise of the eigenvalue matrix $\lambda_1^{[2]}$. For case 2, the orders of perturbation between $\tilde{D}$ and D, as well as $\tilde{K}$ and K, increase one and two, respectively, provided the noise of the eigenvector matrix is perturbed by 10^{-4}.

VI. Conclusions

We have developed an efficient algorithm to incorporate the measured experimental model data into a QEP of an analytical finite element model so that the corrected finite element model closely matches the experimental data. The cost function J_2 of problem (9) in a least-squares sense is chosen by the penalty function given by Friswell et al.,[10] which is slightly different from J_1 proposed by Mottershead and Friswell.[4] Our algorithm is different from the algorithm developed in Ref. 4, which needs to solve a generalized algebraic Riccati equation and in which the resulting solutions are necessary but not sufficient to $\min\{J_1\}$. The solution for $\min\{J_2\}$ computed by the new proposed algorithm is necessary and sufficient to optimization problem (9), which requires only $\mathcal{O}(nm^2)$ FLOPS, with m being the number of experimentally measured modes.

References

[1] Friswell, M. I., and Mottershead, J. E., *Finite Element Model Updating in Structural Dynamics,* Kluwer Academic, Dordrecht, The Netherlands, 1995.

[2] Srinathkumr, S., "Eigenvalue/Eigenvector Assignment Using Output Feedback," *IEEE Transactions on Automatic Control*, Vol. AC-23, No. 1, 1978, pp. 79–81.

[3] Mottershead, J. E., and Friswell, M. I., "Model Updating in Structural Dynamics: A Survey," *Journal of Sound and Vibration*, Vol. 167, No. 2, 1993, pp. 347–375.

[4] Zimmerman, D., and Widengren, M., "Correcting Finite Element Models Using a Symmetric Eigenstructure Assignment Technique," *AIAA Journal*, Vol. 28, No. 9, 1990, pp. 1670–1676.

[5] Minas, C., and Inman, D. J., "Correcting Finite Element Models with Measured Modal Results Using Eigenstructure Assignment Methods," *Proceedings of the 4th IMAC Conference*, Union College, Schenectady, NY, 1987, pp. 583–587.

[6] Datta, B. N., "Finite Element Model Updating, Eigenstructure Assignment and Eigenvalue Embedding Techniques for Vibrating Systems," *Mechanical Systems and Signal Processing*, Special Volume on Vibration Control, Vol. 16, No. 1, 2002, pp. 83–96.

[7] Datta, B. N., Elhay, S., Ram, Y. M., and Sarkissian, D. R., "Partial Eigenstructure Assignment for the Quadratic Pencil," *Journal of Sound and Vibration*, Vol. 230, No. 1, 2000, pp. 101–110.

[8] Datta, B. N., and Sarkissian, D. R., "Theory and Computations of Some Inverse Eigenvalue Problems for the Quadratic Pencil," *Structured Matrices in Mathematics, Computer Science, and Engineering. I, Contemp. Math. 280*, American Mathematical Society, Providence, RI, 2001, pp. 221–240.

[9] Lin, W. W., and Wang, J. N., "Partial Pole Assignment for the Quadratic Pencil by Output Feedback Control with Feedback Designs," *Num. Lin. Alg. Appl.*, Vol. 11, No. 1, 2004, pp. 41–58.

[10] Friswell, M. I., Inman, D. J., and Pilkey, D. F., "Direct Updating of Damping and Stiffness Matrices," *AIAA Journal*, Vol. 36, No. 3, 1998, pp. 491–493.

[11] Kuo, Y. C., Lin, W. W., and Xu, S. F., "A New Model Updating Method for Quadratic Eigenvalue Problems," NCTS TR 2005-1-004, National Tsing-Hua Univ., Taiwan, Republic of China, http://math.cts.nthu.edu.tw/Mathematics/preprints/prep2005-1-004.pdf [cited 15 January 2005].

[12] Zimmerman, D., and Kaouk, M., "Eigenstructure Assignment Approach for Structural Damsge Detection," *AIAA Journal*, Vol. 30, No. 7, 1992, pp. 1848–1855.

[13] Kuo, Y. C., Lin, W. W., and Xu, S. F., "On a General Solution of Partially Described Inverse Quadratic Eigenvalue Problems and Its Applications," NCTS TR 2005-1-001, National Tsing-Hua Univ., Taiwan, Republic of China, http://math.cts.nthu.edu.tw/Mathematics/preprints/prep2005-1-001.pdf [cited 5 January 2005].

A. Berman
Associate Editor